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Jamming in complex gradient networks

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Flows of physical quantities in large complex networks, natural or man made, rely in general on some scalar gradients existing in the networks. We investigate, analytically and numerically, under what conditions jamming in gradient flows can occur in random and scale-free networks. We find that the degree of jamming typically increases with the average connectivity $\langle k\rangle$ of the network. A crossover phenomenon is uncovered where for $\langle k\rangle < k_c$ (k_c denotes a critical connectivity, estimated to be about 10), scale-free networks have a higher level of congestion than random networks with the same $\langle k\rangle$, while the opposite occurs for $\langle k\rangle > k_c$.

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Studies of complex networks have attracted a great deal of interest since the discoveries of the small-world [1] and scale-free [2,3] properties in many natural and man-made networks [4–7]. In complex networks such as the Internet, the network of financial trades, the neuronal system, the power grid, metabolic network, etc., the flow properties of the transported entities (such as information, energy, chemicals, etc.) become of primary interest. In particular, flow congestion, or *jamming*, and its dynamical relation to network structure has become a topic of recent investigation [8,9]. The detailed mechanism for traffic flow varies from case to case, depending on the particular process in the network under consideration. For instance, in a neural network, flow of information is accomplished by the propagation and firing of electrical pulses. In the Internet, digital information flows according to a set of computer instructions. In a social network, rumor propagates along the routes established based on personal and/or professional relationships among individuals in the network. To be able to consider various networks in a general framework, it is reasonable to hypothesize the existence of a gradient field that governs the information flow on the network.

Recently, in Ref. [8], a general framework was established considering gradients generated flows, and the problem of jamming in the network was addressed [8,9]. In this framework, each node in the network is assigned a random weight, and the transport process is guided by the local gradients at nodes. Their finding is that random networks are more susceptible to jamming than scale-free networks. This observation was based on comparing the evolution of the jamming coefficient for the network *growth processes*, in which new nodes are added to an existing network according to some stochastic rules. In particular, for growing binomial random graphs, the same coefficient goes to the value of maximal congestion with increasing network size, N. The reason for this lies in the fact that the congestion factor depends on only two-step neighborhoods of the nodes [9], and in particular it is determined mainly by the average degree $\langle k\rangle$ of the nodes [9]. Since for the scale-free networks with degree exponents larger than 2, $\langle k\rangle$ becomes independent of N, the congestion does not grow with N, whereas for growing random graphs, $\langle k\rangle$ grows linearly with N leading to maximal jamming. In most realistic scale-free networks documented so far [4–7], the average connectivity is small (less than 10). An interesting issue, then, concerns the jamming properties of random and scale-free networks with same parameters $\langle k\rangle$ and N. Notice that a random network with $\langle k\rangle$ values comparable to those in realistic scale-free networks can be generated, for instance, by the binomial model [5] or by rewiring the links in a sparse, regular network, as suggested by Watts and Strogatz [1].

In this Rapid Communication, we report results from a systematic study of the jamming problem in gradient networks. We consider three different types of networks (regular, random, and scale-free) and focus on how the average connectivity affects the degree of jamming. For a class of regular networks, the degree of jamming can be calculated analytically. For random and scale-free networks, we are able to obtain approximate estimates for the degree of jamming. Our main finding is that there exists a critical connectivity parameter k_c below which ($\langle k\rangle < k_c$) the level of the jamming in a scale-free network is higher than for a random graph with the same $\langle k\rangle$ and N; however, above k_c the opposite occurs. The numerically estimated value of k_c is about 10, which is larger than the values in most scale-free networks studied. The conclusion is then that in realistic situations the level of jamming is higher in scale-free networks than in randomlike graphs with the same average connectivity and size.

Given a network of N nodes, a gradient field can be conveniently established by making the network *weighted*, as follows. We begin by assigning a random weight w_i (drawn from a uniform distribution in [0, 1]) to each node in the network. For node i, we examine all its neighboring nodes (including itself) and identify a node j that has the largest weight. This enables a directed link to be specified from node i to j. A self-linked loop is possible if i has the largest weight among all its neighbors. A gradient network can then be defined as the collection of all directed links [8,9]. Regardless of the nature of the network, e.g., regular, random, or scale-free, the way by which the gradient is established stipulates that there be no loops in the network except self loops. We assume that at a given time, a certain amount of information is generated in the network (e.g., data packets in a computer network) and flows along the direction of the links under the gradient field. Let N_{send} and N_{receive} be the number of nodes sending and receiving information at a certain instant, respectively. Then, in the case when a node receives a packet from exactly one other node, i.e., $N_{\text{receive}} = N_{\text{send}}$, no information is delayed in the network and there is thus no jamming. Jamming (or queue formation) occurs when $N_{\text{receive}} < N_{\text{send}}$. The following jamming factor J can thus be defined to measure the degree of jamming [8]

$$J = 1 - \langle\langle N_{\text{receive}}/N_{\text{send}}\rangle_w\rangle_{\text{network}} = R(0), \qquad (1)$$

where $\langle\cdots\rangle_w$ and $\langle\cdots\rangle_{\text{network}}$ denote the statistical average over realizations of the set of weights and the network configuration, respectively. Note that a node requires at least one outgoing link to send information and at least one incoming link to receive information. In the definition (1), $R(0)$ is the probability that a randomly selected node has no incoming link [8,9], which can be calculated numerically. Apparently, $J=0$ corresponds to free traffic on the network without jamming, and $J=1$ indicates the worst case of jamming, where a negligible number of nodes are processing the flow of the rest.

To analyze and understand the degree of jamming for random networks, we start from the simpler case of a one-dimensional regular network on a ring, where each node has two neighbors and there is one link between any two neighboring nodes, i.e., $\langle k\rangle = 2$. A small-world network can be obtained by rewiring a few links [1]. Rewiring all existing links leads to a completely random network. In general, random networks with large $\langle k\rangle$ values can be generated by the binomial model [5], where each pair of nodes is linked with probability $R = \langle k\rangle/N$. To generate a scale-free network with $\langle k\rangle = 2$, we use the standard *BA* model [2,3]. First consider the regular network. For two neighboring nodes A and B, the direction of the (gradient) link can be from A to B or from B to A. There can also be a self loop at A or B. A link may also disappear with respect to the gradient field. We say a node has an incoming link if there is a gradient link pointing to it. Given node i, there are five possible cases where it has no incoming link, as shown in Fig. 1. Consider the first case shown in Fig. 1(a). This occurs for $w_i < w_{i-1} < w_{i+1}$, the probability of which is $P_{(a)} = 1/6$. The case shown in Fig. 1(b) happens for $w_i < w_{i+1} < w_{i-1}$ and we have $P_{(b)} = 1/6$. Similarly, the case shown in Fig. 1(c) occurs for $w_{i-1} < w_i < w_{i+1} < w_{i-2}$ or $w_{i-1} < w_i < w_{i-2} < w_{i+1}$ where, although w_i is larger than w_{i-1}, w_{i-2} is greater than w_i and thus the link between i and $i-1$ disappears in the gradient network. We

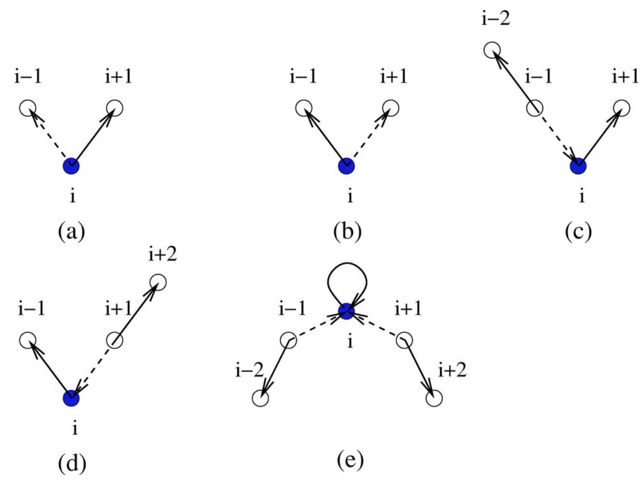

FIG. 1. For the one-dimensional regular network with $\langle k\rangle = 2$, all possible cases for node i to have no incoming links.

thus have $P_{(c)} = 1/4! + 1/4! = 1/12$. The case shown in Fig. 1(d) is for $w_{i+1} < w_i < w_{i-1} < w_{i+2}$ or $w_{i+1} < w_i < w_{i+2} < w_{i-1}$, giving $P_{(d)} = 1/12$. The situation shown in Fig. 1(e) arises when the random weights satisfy one of the following conditions: $w_{i+1} < w_{i-1} < w_i < w_{i-2} < w_{i+2}$, $w_{i+1} < w_{i-1} < w_i < w_{i+2} < w_{i-2}$, $w_{i-1} < w_{i+1} < w_i < w_{i-2} < w_{i+2}$, $w_{i-1} < w_{i+1} < w_i < w_{i+2} < w_{i-2}$. We get $P_{(e)} = 1/5! + 1/5! + 1/5! + 1/5! = 1/30$. The jamming factor is thus given by $J = P_{(a)} + P_{(b)} + P_{(c)} + P_{(d)} + P_{(e)} = 8/15 \simeq 0.533$. In fact, for the one-dimensional Kth power of a ring, it is possible to obtain the analytic dependence of the jamming factor on the connectivity $\langle k\rangle = 2K$ [10]. By using recursive integrals [9] it can be shown that the jamming will increase with increasing average degree, as $J \simeq 1 - 2/\langle k\rangle$ [10].

To estimate, for a random or scale-free network, the probability P that a randomly chosen node has no incoming gradient links, we consider a number of local structures of the network containing the chosen node (drawn as a solid circle in Fig. 2). Figure 2 shows only 16 local structures among many other ones, where all nearest-neighbor nodes of the chosen node have links less than or equal to 4. In this figure, diamonds and open circles mean a node with more than one

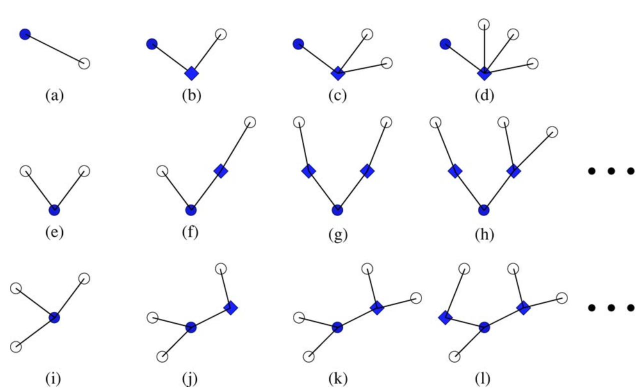

FIG. 2. All possible cases for a selected node (solid circle) to have no incoming links. Diamonds denote nodes with more than one link.

TABLE I. Value of jamming factor when 0%, 20%,…,80% of links in the regular network with $\langle k\rangle=2$ are rewired. The total number of nodes is $N=10\,000$.

%	0	20	40	60	80
J	0.5332	0.5498	0.5631	0.5677	0.5770

link and a node with only one link, respectively. For making the estimate, we only present treelike local configurations, with no loops. This will give an estimate to a lower bound for P, since the existence of loops will further increase its value. For example, in Fig. 2(e) the value of P for the configuration is 1/3, while the triangular configuration formed from this by connecting the two empty circles with a link has $P=2/3$. Since the probability for two empty circles to be connected is $\langle k\rangle/N$, the small increase in P is about $\langle k\rangle/3N$ for the triangular configuration, which does not affect our conclusion. The probabilities for the cases shown in Figs. 2(a)–2(l) are given by $P_{(a)}=\frac{1}{2}$, $P_{(b)}=\frac{1}{2}+\frac{1}{6}$, $P_{(c)}=\frac{1}{2}+\frac{1}{4}$, $P_{(d)}=\frac{1}{2}+\frac{3}{10}$, $P_{(e)}=\frac{1}{3}$, $P_{(f)}=\frac{1}{3}+\frac{1}{12}\times 2$, $P_{(g)}\simeq 0.3668$, $P_{(h)}\simeq 0.4335$, $P_{(i)}=1/4$, $P_{(j)}\simeq 0.3997$, $P_{(k)}\simeq 0.4507$, and $P_{(l)}\simeq 0.5220$, respectively. The probabilities for all other structures can be calculated in a similar fashion (not shown). In general, P increases with the connectivity of the nearest neighbors of the selected node and the probability that a node with zero or one link has no incoming gradient link is rather large, close unity. By the rewiring process in regular network with $\langle k\rangle=2$, some nodes lose one or two links but some other nodes gain links. Both cases lead in average to the increase of P. Thus we expect that the random network to have larger jamming factor than the regular network. In Table I, we list the numerical values of the jamming factor as the network becomes more random.

To obtain the jamming factor for networks with small values of the average connectivity (say $\langle k\rangle<10$), it is necessary to measure the probabilities for a randomly selected node to have no incoming gradient links, for cases where the chosen node has degree one, two, or three on the substrate network. The results for $\langle k\rangle=2$ and $N=10\,000$ are summarized in Table II, where the random network was generated by the binomial model [5].

With the information provided in Table II, we can now estimate the jamming factor J for random network. If a selected node has only one link, as shown in Figs. 2(a)–2(d), the jamming factor $J(1)$, where $J(l)$ means the probability for a node with l links to have no incoming link in the network, is given by

$$J(1)=\{P_R(1)[\tfrac{1}{2}\widetilde{P}_R(1)+(\tfrac{1}{6}+\tfrac{1}{2})\widetilde{P}_R(2)+(\tfrac{1}{4}+\tfrac{1}{2})\widetilde{P}_R(3)+(\tfrac{3}{10}+\tfrac{1}{2})\widetilde{P}_R(4)\cdots]\}>P_R(1)Q_R(1), \quad (2)$$

where

$$Q_R(1)=\tfrac{1}{2}+\tfrac{1}{6}\widetilde{P}_R(2)+\tfrac{1}{4}[1-\widetilde{P}_R(1)-\widetilde{P}_R(2)-\widetilde{P}_R(3)] \approx 0.6596,$$

$P_R(l)$ is the probability for a node m to have l links, $\widetilde{P}_R(l)=P_R(l)/[1-P_R(0)]$, and the normalization condition $1=\widetilde{P}_R(1)+\widetilde{P}_R(2)+\widetilde{P}_R(3)+\widetilde{P}_R(4)+\cdots$ is used. Similar calculations yield $J(2)\approx P_R(2)Q_R(2)$ and $J(3)\approx P_R(3)Q_R(3)$, where $Q_R(2)\approx 0.4804$ and $Q_R(3)\approx 0.3937$. We numerically observe that about 84% of nodes have less than four links. For $l\geqslant 4$, it is reasonable to assume that the probability for a selected node to have no incoming link is about 0.4. Then, we obtain $J_R\approx J(0)+J(1)+J(2)+J(3)+0.1567\times 0.4\approx 0.5748$.

For scale-free network, which is generated by the *BA* model [2,3] of the same parameters, $\langle k\rangle$ and N, the jamming factor is given by

$$J(1)=\{P_{SF}(1)[\tfrac{1}{2}\times 0+(\tfrac{1}{6}+\tfrac{1}{2})\widetilde{P}_{SF}(2)+(\tfrac{1}{4}+\tfrac{1}{2})\widetilde{P}_{SF}(3)+(\tfrac{3}{10}+\tfrac{1}{2})\widetilde{P}_{SF}(4)+\cdots]\}>P_{SF}(1)Q_{SF}(1), \quad (3)$$

where

$$Q_{SF}(1)=\tfrac{1}{2}+\tfrac{1}{6}\widetilde{P}_{SF}(2)+\tfrac{1}{4}\widetilde{P}_{SF}(3)+\tfrac{3}{10}[1-\widetilde{P}_{SF}(2)-\widetilde{P}_{SF}(3)] \approx 0.7236,$$

$P_{SF}(l)$ is the probability for a node m to have l links in the scale-free network, and $\widetilde{P}_{SF}(l)=P_{SF}(l)/[1-P_{SF}(1)]$. For a scale-free network, nodes connected to a node with only one link typically have more than one link. Thus we have $P_{SF}(1)=0$. Similarly, we have $J(2)\approx P_{SF}(2)Q_{SF}(2)$ and $J(3)\approx P_{SF}(3)Q_{SF}(3)$, where $Q_{SF}(2)\approx 0.4080$ and $Q_{SF}(3)\approx 0.3211$. We observe that in the scale-free network, about 90% of nodes have less than four links. We also observe numerically that the probability that more than 99% of the nodes have no incoming links is slightly greater than 0.2, which can be taken as a conservative estimate of the probability that a selected node has no incoming link. We thus obtain, for a scale-free network,

$$J_{SF}\approx J(1)+J(2)+J(3)+0.0982\times 0.2\approx 0.5936>J_R. \quad (4)$$

Our estimates thus show that for random and scale-free network with same values of $\langle k\rangle$ (small) and N, the jamming factor for the latter is somewhat higher, indicating that scale-free networks have a higher level of congestion. In fact, for $\langle k\rangle=2$, the nearest neighbors of the selected node generally have more types of links in scale-free network than in a random network. Thus, we expect the actual value of $\Delta J\equiv J_{SF}-J_R$ to be larger than our heuristic estimate above. That is, our estimate of the value of ΔJ is only a *lower bound* for the actual value, thereby strengthening our main result that

TABLE II. The probabilities for each node to have degree one, two, three, etc., in the random network and SF network with system size $L=10\,000$ and average connectivity $\langle k\rangle=2$.

The number of link	Random network	SF network
0	0.1511	0.0
1	0.2509	0.6713
2	0.2619	0.1631
3	0.1794	0.0674
4	0.0969	0.0307
…	…	…

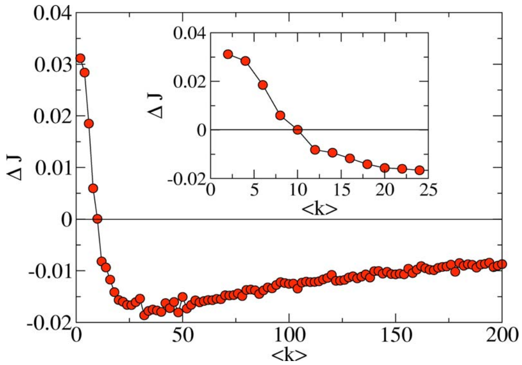

FIG. 3. $\Delta J = J_{SF} - J_R$ between scale-free and random network of 10 000 nodes vs $\langle k \rangle$ for $2 \leq \langle k \rangle \leq 200$. Inset: ΔJ for $2 \leq \langle k \rangle < 25$.

the level of jamming is larger in gradient scale-free networks. Indeed, for $\langle k \rangle = 2$ and $N = 10\,000$, our direct numerical computations give $J_{SF} \approx 0.6830$ and $J_R \approx 0.6519$.

What about the relative values of the jamming factor for random and scale-free networks in the case of large average connectivity $\langle k \rangle$? In this case, we observe numerically (e.g., for $\langle k \rangle \geq 60$ and $N = 10\,000$), more than 90% of nodes in the network have no incoming links. When the neighboring nodes of a chosen node have larger connectivities, the probability for the chosen node to have no incoming links is also larger. In a scale-free network, large connectivity tends to focus on a small set of nodes, in contrast to a random network where nodes with relatively large connectivity are distributed more uniformly. Thus, comparing with scale-free networks, nodes in random networks have more neighbors with large values of k. Another factor is that $Q_{SF}(\langle k \rangle / 2)$ has smaller value than $Q_R(\langle k \rangle / 2)$ for not so small $\langle k \rangle$ (e.g., for $\langle k \rangle > 10$). Thus, intuitively, a random network can have larger jamming factor comparing to a scale-free network with identical values of $\langle k \rangle$ (large) and N. Numerical computation indeed provides evidence supporting this intuition, as shown in Fig. 3; the difference ΔJ as a function of the average connectivity $\langle k \rangle$ for $2 \leq \langle k \rangle \leq 200$ (the inset shows the same plot but for $2 \leq \langle k \rangle < 25$). We see that ΔJ is positive for small $\langle k \rangle$ but it is negative for large $\langle k \rangle$ values, and a crossover occurs for a critical value $\langle k \rangle_c \approx 10$.

Recently two models for traffic congestion on complex networks have been reported [11]. In these models the capacity of packet delivery of each node is given by βB, where β is a capacity parameter. In the first model B is the number of links of a node, while in the second model B is the number of shortest path passing through a node. It was found that in the first model scale-free networks are more jammed than random networks with the same size and the average connectivity. But in the second model, scale-free networks are less jammed than random networks for large values of β. In general, the dynamics of traffic flow on complex networks are affected by various factors such as gradient flow, capacity of a node for packet delivery, and the network structure, etc. In a complex network, the failure of a few nodes may have some serious effects. For instance, disintegration of the network may result from the process of cascading failures [12]. How gradient flows on complex networks are affected by failures of a small number of nodes (e.g., as caused by intentional attacks) is an interesting but open question.

In summary, our analytic estimates and numerical computations of the jamming factor for information flow in gradient networks suggest that the average network connectivity plays an important role in determining the susceptibility of scale-free networks to jamming as compared with random networks. For networks where the average connectivity is small, scale-free networks are more prone to jamming than random networks with the same average connectivity. Since most realistic networks have connectivities that fall in our "small" regime, our result should be relevant. To ensure free information flow in complex networks is important and of broad interest to a variety of disciplines. We hope our results here can be useful for understanding how jamming occurs and for devising strategies to minimize jamming in complex networks.

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[1] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[2] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[3] A.-L. Barabási, R. Albert, and H. Jeong, Physica A **272**, 173 (1999).
[4] S. H. Strogatz, Nature (London) **410**, 268 (2001).
[5] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[6] M. E. J. Newman, SIAM Rev. **45**, 167 (2003).
[7] D. J. Watts, *Small Worlds* (Princeton University Press, Princeton, 1999); J. F. F. Mendes, S. N. Dorogovtsev, and A. F. Ioffe, *Evolution of Networks: From Biological Nets to the Internet and the WWW* (Oxford University Press, Oxford, 2003); R. Pastor-Satorras and A. Vespignani, *Evoloution and Structure of the Internet: A Statistical Physics Approach* (Cambridge University Press, Cambridge, 2004).
[8] Z. Toroczkai and K. E. Bassler, Nature (London) **428**, 716 (2004).
[9] Z. Toroczkai, B. Kozma, K. E. Bassler, N. W. Hengartner, and G. Korniss, e-print cond-mat/0408262.
[10] Z. Toroczkai (private communication).
[11] L. Zhao, Y.-C. Lai, K. Park, and N. Ye, Phys. Rev. E **71**, 026125 (2005).
[12] L. Zhao, K. Park, and Y.-C. Lai, Phys. Rev. E **70**, 035101(R) (2004).